On the Creation of Scalar Particles in a Flat Robertson-Walker Space-time

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The problem of particle creation from vacuum in a flat Robertson-Walker space-time is studied. Two sets of exact solutions for the Klein Gordon equation are given when the scale factor is $a^2(\eta) = a + b\tanh(\lambda\eta) + c\tanh^2(\lambda\eta)$. Then the canonical method based on Bogoliubov transformation is applied to calculate the pair creation probability and the density number of created particles. Some particular cosmological models such as radiation dominated universe and Milne universe are discussed. For both cases the vacuum to vacuum transition probability is calculated and the imaginary part of the effective action is extracted.

I. INTRODUCTION

As we know there is no well defined theory of quantum gravity and the quantization of gravitational fields is one of unsolved problems in theoretical physics. Therefore the straightforward way to treat gravitational effects is the quantum field theory in curved space-time which proved most fruitful in describing interaction of matter with gravity and finding physically meaningful quantities [1–5]. The quantum field theory in a curved space is the first approximation of quantum gravity where the gravitational field described by the metric of the space-time is treated as classical field and the matter is described by quantum fields which propagate in such a gravitational field. It is well known also that the most significant prediction of this theory is the phenomenon of particle creation from the vacuum. Generally in curved space-time it is difficult to define a physical vacuum state for the quantum field and when this vacuum state is defined in the remote past it is usually unstable so that it may differ from the vacuum state in the remote future, and spontaneous creation of particles occurs .

The importance of particle production by gravitational fields comes from the diversity of its applications in contemporary cosmology; Particle creation effects could have consequences for early universe cosmology and could play a role in the exit from inflationary universe and in the cosmic evolution [6–10]. The application of this phenomenon in black hole physics, such as Hawking radiation, is also well known [11].

In order to study the phenomenon of particle creation in gravitational fields we found in literature several techniques such as the adiabatic approach [12–14], the Hamiltonian diagonalization method [15–20], the Green function approach [21, 22], the Feynman path integral technique [23, 24], the "in" and "out" states formalism [25–28] as well as the semiclassical

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WKB approximation [29–33]. In addition we cite the technique based on vacuum to vacuum transition amplitude and Schwinger-like effective action [34, 35].

The aim of this paper is to study the creation of scalar particles in a flat Robertson-Walker space-time provided with metric of the form

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right) \tag{1}$$

In terms of conformal time $\eta = \int dt/a(t)$ this metric reads

$$ds^2 = C(\eta)\left[d\eta^2 - dx^2 - dy^2 - dz^2\right] \tag{2}$$

where the new scale factor $C(\eta)$ is defined from $a(t)$ as follows $C(\eta) = \tilde{a}^2(\eta) \equiv a^2[t(\eta)]$. We choose for the scale factor the form

$$C(\eta) = a + b\tanh(\lambda\eta) + c\tanh^2(\lambda\eta) \tag{3}$$

where a, b and c are positive parameters.

We can see that this form is the generalization of various particular cases found in literature; When $c = 0$, we have a cosmological model with $a^2(\eta) = a + b\tanh(\lambda\eta)$ which has been widely studied [36–38]. With a particular choice of parameters a, b and c we get some models discussed in [39, 40]. In addition, this universe becomes a radiation dominated one when $a = b = 0$, $c = \frac{a_0^4}{4\lambda^2}$ and $\lambda \to 0$. We can also make connection with a Milne universe (i.e. $a(t) = a_1 t$.) when $c = 0$, $\lambda = a_1$, $b = a = \frac{a_1^2}{2\varepsilon}$ by making the change $\eta \to \eta + \frac{\ln\varepsilon}{2\lambda}$ and taking the limit $\varepsilon \longrightarrow 0$.

In the first stage we introduce a scalar field propagating in Robertson-Walker space-time and we give two sets of exact solutions for the Klein Gordon equation. Next we use the relation between these two sets to determine the probability of pair creation, the density number of created particles and the vacuum persistence. Finally we discuss some particular examples.

II. SCALAR FIELD AND KLEIN GORDON EQUATION

Let us consider a scalar matter field Φ with mass m subjected to the gravitational field described by the metric $g_{\mu\nu}$. The dynamics of this system is in general governed by the action

$$S = \int d^4x \left[\sqrt{-g}\left(-g^{\mu\nu}\partial_\mu\Phi\partial_\nu^*\Phi - m^2\Phi\Phi^* - V(g)\Phi\Phi^*\right)\right] \tag{4}$$

where $V(g)$ is a function of invariant combinations of the metric tensor $g_{\mu\nu}$ and its partial derivatives [41, 42]

$$V(g) = \xi_1 R + \xi_2 R_{GB}^2, \tag{5}$$

where R is the Ricci scalar, ξ_1 and ξ_2 are two numerical factors and R_{GB}^2 is a scalar which explains the Gauss-Bonnet coupling

$$R_{GB}^2 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{6}$$

In such a case the Klein Gordon equation can be written in the form

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\Phi\right) + \left(m^2 + V(g)\right)\Phi = 0. \tag{7}$$

In the present work we interest only on conformally coupled scalar field (i.e. $\xi_1 = \frac{1}{6}$ and $\xi_2 = 0$). If we choose to work with conformal time η which is convenient to the present coupling and we introduce a new field $\psi(x)$ so that

$$\Phi(x) = \frac{1}{\sqrt{C(\eta)}}\psi(x) = \frac{1}{\sqrt{C(\eta)}}\chi(\vec{x})\varphi(\eta), \tag{8}$$

where $\chi(\vec{x})$ has, in the case of flat space-time, the form of a plane wave $\chi(\vec{x}) \sim \exp\left(i\vec{k}.\vec{r}\right)$, we can obtain the simplified equation

$$\left[\frac{d^2}{d\eta^2} + \omega^2(\eta)\right]\varphi(\eta) = 0, \tag{9}$$

with

$$\omega^2(\eta) = k^2 + m^2 C(\eta). \tag{10}$$

Since equation (9) is of second order there are only two independent solutions and all other solutions can be expressed in terms of these two independent ones. Here we want to find two sets of independent solutions so that the two functions $\varphi_{in}^\pm$ of the first set behave like positive and negative energy states at $\eta \to -\infty$ and the two functions $\varphi_{out}^\pm$ of the second set behave like positive and negative energy states at $\eta \to +\infty$. Therefore, before to look for exact solutions for equation (9), let us first, examine their behavior when $\eta \to \pm\infty$. We easily obtain

$$\varphi_{in}^\epsilon(\eta) = \exp\left(-i\epsilon\omega_{in}\eta\right) \tag{11}$$
$$\varphi_{out}^\epsilon(\eta) = \exp\left(-i\epsilon\omega_{out}\eta\right), \tag{12}$$

where ϵ indicates the positive or the the negative frequency mode ($\epsilon = \pm 1$) and ω_{out} and ω_{in} are given by

$$\omega_{in} = \sqrt{k^2 + m^2(a + c - b)} \tag{13}$$
$$\omega_{out} = \sqrt{k^2 + m^2(a + c + b)}. \tag{14}$$

Note that behaviors in (11) and (12) are the same as semi-classical solutions obtained from Hamilton-Jacobi equation because the space-time is asymtotically Minkowskian.

Now in order to solve equation (9) we make the change $\eta \to \xi$, where

$$\xi = \frac{1 + \tanh(\lambda\eta)}{2}. \tag{15}$$

The resulting equation that takes the form

$$\left[\frac{\partial^2}{\partial\xi^2} + \left(\frac{1}{\xi} - \frac{1}{1-\xi}\right)\frac{\partial}{\partial\xi} + \left(\frac{\omega_{in}^2}{4\lambda^2}\frac{1}{\xi} - \frac{m^2 c}{\lambda^2}\right.\right.$$
$$\left.\left. + \frac{\omega_{out}^2}{4\lambda^2}\frac{1}{(1-\xi)}\right)\frac{1}{\xi(1-\xi)}\right]\tilde{\varphi}(\xi) = 0 \tag{16}$$

is a Riemann type equation [43]

$$\left[\frac{\partial^2}{\partial\xi^2} + \left(\frac{1-\alpha_1-\alpha_1'}{\xi} - \frac{1-\alpha_3-\alpha_3'}{1-\xi}\right)\frac{\partial}{\partial\xi} + \right.$$
$$\left.\left(\frac{\alpha_1\alpha_1'}{\xi} - \alpha_2\alpha_2' + \frac{\alpha_3\alpha_3'}{1-\xi}\right)\frac{1}{\xi(1-\xi)}\right]\tilde{\varphi}(\xi) = 0 \tag{17}$$

where

$$\begin{aligned}
\alpha_1 &= -\alpha_1' = i\frac{\omega_{in}}{2\lambda} \\
\alpha_3 &= -\alpha_3' = i\frac{\omega_{in}}{2\lambda} \\
\alpha_2 &= 1-\alpha_2' = \frac{1}{2} + \sqrt{\frac{m^2c}{\lambda^2} - \frac{1}{4}},
\end{aligned} \tag{18}$$

with the condition $\alpha_1 + \alpha_1' + \alpha_2 + \alpha_2' + \alpha_3 + \alpha_3' = 1$.

Following [43] we can find for equation (9) several sets of solutions that can be written in terms of hypergeometric functions. Taking into account the behavior of positive and negative energy states we can classify our to sets as follows; for the "in" states we have

$$\tilde{\varphi}_{in}^{+}(\xi) = \frac{1}{\sqrt{2\omega_{in}}}\xi^{-i\frac{\omega_{in}}{2\lambda}}(1-\xi)^{i\frac{\omega_{out}}{2\lambda}}$$
$$F\left(\frac{1}{2} + i\frac{\omega_-}{\lambda} + i\delta, \frac{1}{2} + i\frac{\omega_-}{\lambda} - i\delta; 1 - i\frac{\omega_{in}}{\lambda}; \xi\right) \tag{19}$$

and

$$\tilde{\varphi}_{in}^{-}(\xi) = \frac{1}{\sqrt{2\omega_{in}}}\xi^{i\frac{\omega_{in}}{2\lambda}}(1-\xi)^{-i\frac{\omega_{out}}{2\lambda}}$$
$$F\left(\frac{1}{2} - i\frac{\omega_-}{\lambda} + i\delta, \frac{1}{2} - i\frac{\omega_-}{\lambda} - i\delta; 1 + i\frac{\omega_{in}}{\lambda}; \xi\right), \tag{20}$$

with

$$\omega_\pm = \frac{\omega_{out} \pm \omega_{in}}{2} \tag{21}$$

and

$$\delta = \frac{1}{2}\sqrt{\frac{4m^2c}{\lambda^2} - 1}. \tag{22}$$

The factors $(2\omega_{in})^{-1/2}$ and $(2\omega_{out})^{-1/2}$ are determined by the use of the following normalization condition

$$\varphi_k^*\dot{\varphi}_k - \varphi_k\dot{\varphi}_k^* = 2i, \tag{23}$$

which explains the conservation of the Klein Gordon particle current.

For the "out" states we have

$$\tilde{\varphi}_{out}^{+}(\xi) = \frac{1}{\sqrt{2\omega_{out}}}\xi^{-i\frac{\omega_{in}}{2\lambda}}(1-\xi)^{i\frac{\omega_{out}}{2\lambda}}$$
$$F\left(\frac{1}{2} + i\frac{\omega_-}{\lambda} + i\delta, \frac{1}{2} + i\frac{\omega_-}{\lambda} - i\delta; 1 + i\frac{\omega_{out}}{\lambda}; 1 - \xi\right) \tag{24}$$

and

$$\tilde{\varphi}_{out}^{-}(\xi) = \frac{1}{\sqrt{2\omega_{out}}} \xi^{i\frac{\omega_{in}}{2\lambda}}(1-\xi)^{-i\frac{\omega_{out}}{2\lambda}}$$
$$F\left(\frac{1}{2} - i\frac{\omega_{-}}{\lambda} + i\delta, \frac{1}{2} - i\frac{\omega_{-}}{\lambda} - i\delta; 1 - i\frac{\omega_{out}}{\lambda}; 1 - \xi\right). \tag{25}$$

Having shown how to solve Klein Gordon equation for spinless particle conformally coupled to gravitational field described by the metric (2), let us investigate the particle creation process. In the next section we quantify the scalar matter field and we use the solutions of the field equation to analyze the problem in question.

III.   FIELD QUANTIZATION AND PARTICLE CREATION

In the first stage we write the field operator in it's Fourier decomposition

$$\hat{\psi}(\vec{x}, \eta) = \frac{1}{\sqrt{2}} \int d^3k \left[\varphi_k^*(\eta)\chi_k(\vec{x})\,\hat{a}_k + \varphi_k(\eta)\chi_k^*(\vec{x})\,\hat{b}_k^+\right] \tag{26}$$

where, in canonical quantization formalism, the operators $\hat{a}_k$ and $\hat{b}_k$ verify the following commutation relation

$$\left[\hat{a}_k, \hat{a}_{k'}^+\right] = \left[\hat{b}_k, \hat{b}_{k'}^+\right] = \delta\left(\vec{k} - \vec{k}'\right). \tag{27}$$

With the help of the normalization condition (23) we can find without difficulties the following expression of the Hamiltonian associated with the scalar field system

$$H = \frac{1}{2} \int d^3k \left[E_k(\eta)\left(\hat{a}_k\hat{a}_k^+ + \hat{b}_k^+\hat{b}_k\right) + F_k^*(\eta)\,\hat{b}_k\hat{a}_k + F_k(\eta)\,\hat{a}_k^+\hat{b}_k^+\right] \tag{28}$$

with

$$E_k(\eta) = |\dot{\varphi}_k(\eta)|^2 + \omega_k^2(\eta)|\varphi_k(\eta)|^2 \tag{29}$$
$$F_k(\eta) = \dot{\varphi}_k^2(\eta) + \omega_k^2(\eta)\varphi_k^2(\eta). \tag{30}$$

Here we remark that H is not diagonal at any time. For $\eta \to \pm\infty$, however, $F_k(\eta) = 0$ and H becomes diagonal. In this situation we can define two vacuum states $|0_{in}\rangle$ and $|0_{out}\rangle$. The state $|0_{in}\rangle$ is an initial quantum vacuum state in the remote past with respect to a static observer and $|0_{out}\rangle$ is a final quantum vacuum state in the remote future with respect to the same observer. This gives some vacuum instability which leads to particle creation.

Now, as we mentioned above, in order to determine the probability of pair creation and the density of created particles we use the solutions of the Klein Gordon equation by considering the canonical method based the relation between "in" and "out" states. In other wards Bogoliubov transformation connecting the "in" with the "out" states can be projected in Fock space to be converted into relation between the creation and annihilation operators; therefore, the probability of pair creation and the density of created particles will be given in terms of Bogoliubov coefficients.

So to get connection between "in" and "out" modes let us use the relation between hypergeometric functions [43]

$$F(u, v; w; \xi) = \frac{\Gamma(w)\Gamma(w - v - u)}{\Gamma(w - u)\Gamma(w - v)} F(u, v; u + v - w + 1; 1 - \xi)$$
$$+ (1 - \xi)^{w-u-v} \frac{\Gamma(\gamma)\Gamma(u + v - w)}{\Gamma(u)\Gamma(v)}$$
$$F(w - u, w - v; w - v - u + 1; 1 - \xi) \tag{31}$$

and the property

$$F(u, v; w; \xi) = (1 - \xi)^{w-u-v} F(w - u, w - v; w; \xi) \tag{32}$$

to obtain the so-called Bogoliubov transformation connecting "in" and "out" states

$$\varphi_{in}^{+} = \alpha\varphi_{out}^{+} + \beta\varphi_{out}^{-} \tag{33}$$
$$\varphi_{in}^{-} = \beta^{*}\varphi_{out}^{+} + \alpha^{*}\varphi_{out}^{-}, \tag{34}$$

where the Bogoliubov coefficients α and β are given by

$$\alpha = \sqrt{\frac{\omega_{out}}{\omega_{in}}} \frac{\Gamma\left(1 - i\frac{\omega_{in}}{\lambda}\right)\Gamma\left(-i\frac{\omega_{out}}{\lambda}\right)}{\Gamma\left(\frac{1}{2} - i\frac{\omega_{+}}{\lambda} - i\frac{1}{2}\sqrt{\frac{4m^2c}{\lambda^2} - 1}\right)\Gamma\left(\frac{1}{2} - i\frac{\omega_{+}}{\lambda} + i\frac{1}{2}\sqrt{\frac{4m^2c}{\lambda^2} - 1}\right)} \tag{35}$$

and

$$\beta = \sqrt{\frac{\omega_{out}}{\omega_{in}}} \frac{\Gamma\left(1 - i\frac{\omega_{in}}{\lambda}\right)\Gamma\left(i\frac{\omega_{out}}{\lambda}\right)}{\Gamma\left(\frac{1}{2} + i\frac{\omega_{-}}{\lambda} - i\frac{1}{2}\sqrt{\frac{4m^2c}{\lambda^2} - 1}\right)\Gamma\left(\frac{1}{2} + i\frac{\omega_{-}}{\lambda} + i\frac{1}{2}\sqrt{\frac{4m^2c}{\lambda^2} - 1}\right)} \tag{36}$$

with $|\alpha|^2 - |\beta|^2 = 1$. The relation between the creation and annihilation operators is then

$$a_{out} = \alpha \, a_{in} + \beta^{*} b_{in}^{+} \tag{37}$$
$$b_{out}^{+} = \beta \, a_{in} + \alpha^{*} b_{in}^{+}. \tag{38}$$

For the process of particle creation the probability amplitude that we want to calculate is defined by

$$\mathcal{A} = \langle 0_{out} | a_{out} b_{out} | 0_{in} \rangle. \tag{39}$$

Taking into account that

$$b_{out} = \frac{1}{\alpha^{*}} b_{in} + \frac{\beta^{*}}{\alpha^{*}} a_{out}^{+} \tag{40}$$

we obtain

$$\mathcal{A} = \langle 0_{out} | a_{out} b_{out} | 0_{in} \rangle = \frac{\beta^{*}}{\alpha^{*}} \langle 0_{out} | 0_{in} \rangle. \tag{41}$$

The probability to create one pair of particles from vacuum is then

$$\mathcal{P}_{creat.}(k) = \left|\frac{\beta^*}{\alpha^*}\right|^2. \tag{42}$$

Using the following properties of the Gamma functions [43]

$$\Gamma(z+1) = z\Gamma(z), \tag{43}$$

$$|\Gamma(ix)|^2 = \frac{\pi}{x \sinh \pi x} \tag{44}$$

and

$$\left|\Gamma(\frac{1}{2} + ix)\right|^2 = \frac{\pi}{\cosh \pi x} \tag{45}$$

we arrive at

$$\mathcal{P}_{creat.}(k) = \frac{\cosh\left(2\pi\frac{\omega_-}{\lambda}\right) + \cosh\left(2\pi\delta\right)}{\cosh\left(2\pi\frac{\omega_+}{\lambda}\right) + \cosh\left(2\pi\delta\right)}. \tag{46}$$

Let $\mathcal{C}_k$ to be the probability to have no pair creation in the state k. The quantity $\mathcal{C}_k \left(P_{creat.}\right)^n$ is then the probability to have only n pairs in the state k. We have

$$\mathcal{C}_k + \mathcal{C}_k P_{creat.} + \mathcal{C}_k \left(P_{creat.}\right)^2 + \mathcal{C}_k \left(P_{creat.}\right)^3 + ... = 1 \tag{47}$$

or simply

$$\mathcal{C}_k = 1 - P_{creat.}. \tag{48}$$

Being aware of $\left|\frac{\beta^*}{\alpha^*}\right|^2 + \left|\frac{1}{\alpha^*}\right|^2 = 1$, we can find the vacuum persistence which reads

$$\mathcal{C}_k = \left|\frac{1}{\alpha^*}\right|^2$$

$$= \frac{\cosh\left(2\pi\frac{\omega_+}{\lambda}\right) - \cosh\left(2\pi\frac{\omega_-}{\lambda}\right)}{\cosh\left(2\pi\frac{\omega_+}{\lambda}\right) + \cosh\left(2\pi\delta\right)}.$$

Another important result is the density number of created particles

$$n(k) = \left\langle 0_{in} \left| a_{out}^+ a_{out} \right| 0_{in} \right\rangle = |\beta|^2 \tag{49}$$

A simple calculation gives

$$n(k) = \frac{\cosh\left(2\pi\frac{\omega_-}{\lambda}\right) + \cosh\left(\pi\sqrt{\frac{4m^2c}{\lambda^2} - 1}\right)}{\cosh\left(2\pi\frac{\omega_+}{\lambda}\right) - \cosh\left(2\pi\frac{\omega_-}{\lambda}\right)}. \tag{50}$$

Here we note that the density number of created particles can be written as

$$n(k) = \frac{1}{\left|\frac{\alpha}{\beta}\right|^2 - 1}, \tag{51}$$

and for large frequencies $n(k)$ becomes a thermal Bose-Einstein distribution

$$n(k) = \frac{1}{\exp\left(\frac{2\pi}{\lambda}\omega_{in}\right) - 1}. \tag{52}$$

From equation (50) one can see that the pair creation process is more important for low impulsion and $n(k) \to 0$ when $k \to \infty$ (i.e. $k >> 1$). Also, for $a - b + c \neq 0$ and $a + b > c$, it is clear that $n(k) \to 0$ when $m >> 1$.

IV.   SPECIAL CASES

In order to illustrate our calculation let us in this section discuss some particular cosmological models such as radiation dominated universe and the Milne universe which are of interest and have been studied by other authors and by using other techniques. We start by radiation dominated universe.

A.   Radiation dominated universe

Considering the case when $a = b = 0$ and $c = \frac{a_0^4}{4\lambda^2}$ and taking the limit $\lambda \to 0$, we make connection with the well-known phase of radiation dominated universe

$$C(\eta) = \frac{a_0^4}{4}\eta^2 \tag{53}$$

and

$$a(t) = a_0\sqrt{t}. \tag{54}$$

In this case the probability to create a pair of particles will be

$$\mathcal{P}_{rad.}(k) = \lim_{\lambda \to 0} \frac{\exp\left(\pi\sqrt{\frac{4m^2c}{\lambda^2} - 1} - \frac{2\pi\omega}{\lambda}\right)}{1 + \exp\left(\pi\sqrt{\frac{4m^2c}{\lambda^2} - 1} - \frac{2\pi\omega}{\lambda}\right)}. \tag{55}$$

Taking into account that

$$\pi\sqrt{\frac{4m^2c}{\lambda^2} - 1} - \frac{2\pi\omega}{\lambda} = m\frac{a_0^2}{2\lambda^2}\sqrt{1 + \frac{4\lambda^2 k^2}{m^2 a_0^4}} - \frac{1}{2}m\frac{a_0^2}{\lambda^2}\sqrt{1 - \frac{\lambda^4}{a_0^4 m^2}} \approx \frac{k^2}{m a_0^2} \tag{56}$$

we get

$$\mathcal{P}_{rad.}(k) = \frac{\exp\left[-\pi\frac{2k^2}{m a_0^2}\right]}{1 + \exp\left[-\pi\frac{2k^2}{m a_0^2}\right]}. \tag{57}$$

The density number of created particles will be

$$n(k) = \exp\left(-\frac{2\pi k^2}{m a_0^2}\right). \tag{58}$$

This results coincide exactly with those found in literature [23].

For the vacuum to vacuum probability we can write

$$
\begin{aligned}
\mathcal{P}_{vac \to vac} &= \exp\left(-2\operatorname{Im} S_{eff}\right) \\
&= \prod_k \mathcal{C}_k \\
&= \prod_k \exp\left[-\ln\left(1+\sigma\right)\right] \\
&= \exp\left[-\sum_k \ln\left(1+\sigma\right)\right],
\end{aligned}
\tag{59}
$$

where $\sigma = \exp\left(-\frac{2\pi k^2}{m a_0^2}\right)$. Consequently, we have

$$
2\operatorname{Im} S_{eff} = \sum_k \ln\left(1+\sigma\right).
\tag{60}
$$

Expanding the quantity $\ln(1+\sigma)$, we get

$$
2\operatorname{Im} S_{eff} = \int \frac{d^3k}{(2\pi)^3} \sum_{n=1} \frac{(-1)^{n+1}}{n} \exp\left(-n\pi \frac{2k^2}{a_0^2 m}\right).
\tag{61}
$$

By doing integration over momentum k

$$
2\operatorname{Im} S_{eff} = \left(\frac{a_0^2 m}{8\pi^2}\right)^{\frac{3}{2}} \sum_{n=1} \frac{(-1)^{n+1}}{n^{\frac{5}{2}}}
\tag{62}
$$

and summing over n, we obtain the following result

$$
2\operatorname{Im} S_{eff} = 0.867 \left(\frac{m a_0^2}{8\pi^2}\right)^{\frac{3}{2}}.
\tag{63}
$$

It is clear that the effect becomes important as soon as the factor a_0 approaches the critical value

$$
a_{cr}^2 \sim \frac{8\pi^2}{m}
\tag{64}
$$

and the pair production rate is of order of $\left(\frac{a_0}{a_{cr}}\right)^3$.

B.   Milne universe

Now we consider the example of Milne universe defined by the scale factor $a\left(t\right) = a_1 t$. In term of conformal time this scale factor reads

$$
C\left(\eta\right) = a^2\left(\eta\right) = a_1^2 \exp\left(2a_1\eta\right).
\tag{65}
$$

As is mentioned above this case may be obtained from (3) by considering the case $c = 0$, $\lambda = a_1$, $b = a = \frac{a_1^2}{2\varepsilon}$, making the change $\eta \to \eta + \frac{\ln \varepsilon}{2\lambda}$ and taking the limit $\varepsilon \longrightarrow 0$. In these conditions the probability to have one pair creation will be

$$\mathcal{P}_{creat.} = \exp\left(-\frac{2\pi}{a_1}k\right), \tag{66}$$

which is in agreement with the same probability calculated by other authors [23].

The vacuum persistence is

$$\mathcal{C}_k = 1 - \exp\left(-\frac{2\pi}{a_1}k\right) \tag{67}$$

and the vacuum to vacuum probability takes the form

$$\exp\left(-2\operatorname{Im} S_{eff}\right) = \prod_k \mathcal{C}_k$$

$$= \exp\left[\sum_k \ln\left(1 - \exp\left(-\frac{2\pi}{a_1}k\right)\right)\right] \tag{68}$$

from which we draw the imaginary part of the effective action

$$2\operatorname{Im} S_{eff} = -\sum_k \ln\left(1 - \exp\left(-\frac{2\pi}{a_1}k\right)\right) \tag{69}$$

Making the Taylor expansion of the logarithm function

$$2\operatorname{Im} S_{eff} = \int \frac{d^3k}{(2\pi)^3} \sum_{n=1} \frac{1}{n}\exp\left(-n\frac{2\pi}{a_1}k\right) \tag{70}$$

and summing over n we obtain

$$2\operatorname{Im} S_{eff} = 1.2 \times \frac{a_1^2}{4\pi^4}. \tag{71}$$

Here we remark that the pair production rate is of order of a_1^2.

C.   Other particular cases

Since the creation of superheavy particles with the mass of the Grand Unification scale in the early Universe has many important cosmological consequences [44–46] we want to discuss in this paragraph cases with $m >> 1$. In general when $m >> 1$, we have shown that $n(k) \to 0$. However, when $a - b + c = 0$ or $c > a + b$ we remark that $n(k) \to C^{st} > 0$ even if $m >> 1$. This implies that it is possible to create superheavy particles in this model.

Let us mention that if we put $c = 0$, we obtain the same results as those found in [37–40] and when $a = \left(\frac{1+\varepsilon}{2}\right)^2$; $b = \frac{1-\varepsilon^2}{2}$; $c = \left(\frac{1-\varepsilon}{2}\right)^2$ our results coincide with those of [39, 40].

V. CONCLUSION

In this paper we have studied the creation of scalar particles in some flat Robertson-Walker space-times. We have considered the canonical method based on Bogoliubov transformation connecting the "in" with the "out" states. We have given two sets of exact solutions for the Klein Gordon field equation and we have used these solutions to expressed the probability of pair creation and the density of created particles in some closed forms. Then we have discussed some particular cosmological models where our results become the same as those obtained by other authors. For the radiation dominated universe and the Milne universe we have calculated the vacuum to vacuum transition probability and we have extracted the nonvanishing imaginary term of the effective action that means that created particles are real and not virtual ones. We have shown also that creation of superheavy particles may be important in the case when $a - b + c = 0$ or $c > a + b$.

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